

A central limit theorem for the Euler integral of a Gaussian random field

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Abstract

Euler integrals of deterministic functions have recently been shown to have a wide variety of possible applications, including in signal processing, data aggregation and network sensing. Adding random noise to these scenarios, as is natural in the majority of applications, leads to a need for statistical analysis, the first step of which requires asymptotic distribution results for estimators. The first such result is provided in this paper, as a central limit theorem for the Euler integral of pure, Gaussian, noise fields.

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1. Introduction

The Euler characteristic $\chi(A)$ of a nice set A is perhaps the oldest, and most fundamental, of its topological invariants. For a compact $A \subset \mathbb{R}^1$, the Euler characteristic is merely the number of its connected components (each one of which will be an interval, possibly containing only a single point). For $A \subset \mathbb{R}^2$, $\chi(A)$ becomes the number of connected components minus the number of holes, while in three dimensions $\chi(A)$ can be written as the alternating sum of the numbers of components, handles and hollows. Similar (and, of course, more precise) definitions as alternating sums of Betti numbers, numbers of facets of simplices of differing dimension (when A is triangulisable) or as indices of critical points when a Morse theoretic setting is appropriate, extend the Euler characteristic to a wide variety of sets in arbitrary dimensions.

However, more important for us is that the Euler characteristic is also a valuation, which means that, when all terms are defined, we have the additivity property,

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B). \quad (1.1)$$

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Given additivity, it is natural to attempt to use χ to define an integral on a suitable family of functions, and, indeed, to a large extent this can be done. The resulting theory is known as Euler integration.

1.1. Euler integration

Although in many ways Euler integration has its roots in classical Integral Geometry, a more complete and modern theory began to evolve in the 1970's. More importantly for us, however, is that it has experienced a rapid development in the past decade from both applied and theoretical aspects, providing for some elegant and novel results. We shall not attempt to survey these here, since the recent papers of Baryshnikov and Ghrist (2009) and Curry et al. (2012) provide excellent and broad expositions. Rather, we shall go directly to two definitions.

Definition 1.1. *Let $M \subset \mathbb{R}^n$ be compact, with finite Euler characteristic. Then a continuous function $f : M \rightarrow \mathbb{R}$ is called tame if the homotopy types of $f^{-1}((-\infty, u])$ and $f^{-1}([u, \infty))$ change only finitely many times as u varies over \mathbb{R} , and the Euler characteristic of each set is finite.*

Definition 1.2. *If $f : M \rightarrow \mathbb{R}$ is tame, then the upper Euler integral of f over M is defined by*

$$\int_M f [d\chi] \triangleq \int_{u=0}^{\infty} [\chi(f > u) - \chi(f \leq -u)] du, \quad (1.2)$$

where

$$\chi(f \leq u) \triangleq \chi(f^{-1}((-\infty, u])),$$

and

$$\chi(f > u) \triangleq \chi(M) - \chi(f \leq u).$$

Reading in between the lines that do *not* appear in the above definition, one would guess that there is also a lower Euler integral (there is!) and that there has to be a more direct way to define an integral that follows from the additivity of (1.1). In fact, this is also true, and, as a result, the Euler integral shares many common properties with the classical theories of integration. However, it is somewhat more delicate, since although (1.1) extends to a finite inclusion-exclusion form, it does not typically extend to the countably infinite case needed for a standard measure based theory of integration. The definition that we have chosen above avoids these issues, and in taking it we follow the lead of Bobrowski and Borman (2012) who, by taking (1.2) as a definition rather than a property, save often irritating but unimportant (for our needs) technicalities.

1.2. A motivating application

An interesting application of the Euler integral is described in Baryshnikov and Ghrist (2009) and Curry et al. (2012).

Suppose that an unknown number of targets are located in a region $M \subset \mathbb{R}^n$, and each target α is represented by its support $U_\alpha \subset M$. Suppose also that the space M is covered with sensors, reporting only the number of targets each one sees. Let $h : X \rightarrow \mathbb{Z}$ be the *sensor field*, i.e.

$$h(x) \triangleq \# \{\text{targets activating the sensor located at } x\}.$$

Then, if all the target supports satisfy $\chi(U_\alpha) = \beta$ for some $\beta \neq 0$, the readings from all the sensors can be combined to obtain the exact number of targets via the relationship

$$N \triangleq \# \{\text{targets}\} = \frac{1}{\beta} \int_M h[d\chi]. \quad (1.3)$$

Note that we do not need to assume anything about the targets other than that they all have the same Euler characteristic. For example, we need not assume that they are all convex or even have the same number of connected components.

While everything in (1.3) is deterministic, Bobrowski and Borman (2012) raises the question as to what happens when the deterministic ‘signal’ $x = \int_M h[d\chi]$, is observed via a noisy measurement $Y = \int_M (h + X)[d\chi]$, where X is a smooth random process on M . They show that, although Euler integrals are not always additive, in this case it is true that

$$Y = \int_M (h + X)[d\chi] = \int_M h[d\chi] + \int_M X[d\chi] = \beta N + \xi,$$

which leads to the obvious estimator \hat{N} of N given by

$$\hat{N} = \beta^{-1} \left(Y - \mathbb{E} \left[\int_M X[d\chi] \right] \right). \quad (1.4)$$

The main result of Bobrowski and Borman (2012) is an elegant calculation of the expectation in (1.4) when X is a smooth Gaussian or Gaussian related random field and M a stratified manifold, based on the Gaussian kinematic formula of Adler and Taylor (2007). Their computation leads to an explicit, closed form (and often somewhat surprising) expression for the expectation. We shall have more to say about this in Section 4.

Motivated by the above, what this paper concentrates on is a central limit theorem (henceforth CLT) needed to go from the estimation provided by \hat{N} to inference.

1.3. A CLT for the Euler integral

The main result of the paper is formulated in Theorem 3.1, which states that if X is a real valued, almost surely C^2 , stationary Gaussian random field on \mathbb{R}^n , satisfying certain technical regularity and decay of memory conditions, and if we define

$$\Psi_{[0,m]^n}[X] \triangleq \int_{[0,m]^n} X[d\chi],$$

then, as $m \rightarrow \infty$,

$$\frac{\Psi_{[0,m]^n}[X] - \mathbb{E}[\Psi_{[0,m]^n}[X]]}{m^{n/2}} \xrightarrow{\mathcal{D}} N(0, \sigma_\Psi^2), \quad (1.5)$$

for some limiting variance $\sigma_\Psi^2 > 0$, where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution.

1.4. Outline of the paper

Before giving the formal version of (1.5) in Section 3, in the following section we shall set up considerable preliminary material, treating the regularity conditions that we require on X as well as background material on Wiener chaos expansions and the Morse theoretical representation of the Euler integral. Then, in Section 3, we apply techniques from Houdré and Pérez-Abreu (1994) and Major (2014) to develop the chaos expansion for the upper Euler integral of a Gaussian random field. This, together with some regularity and convergence results, are combined with a general CLT of Nourdin and Peccati (2012) for chaos expansions, to make up the proof our main CLT. Many of the proofs here owe a lot to the papers by Kratz and Leon (1997), Kratz and León (2001), and especially the recent work of Estrade and León (2015).

In Section 4 we look at a direct calculation of the mean value of Euler integral (for the isotropic case), showing its dependence on the order-one Lipshitz-Killing curvature of M , and discuss the surprising results of Bobrowski and Borman (2012) alluded to above.

1.5. Acknowledgements

1. A reader familiar with the paper Estrade and León (2015) will note a strong resemblance between many of the technical parts of that paper and those in Section 3 in the present paper. This is not accidental, since that paper develops a CLT for the Euler *characteristic* of excursion sets, while we treat a CLT for the Euler *integral*. Neither result implies the other, and, as far as we can tell, neither is derivable from the other without a lot of additional work. Nevertheless, many of the details of the calculations are similar.

We became aware that Estrade and León (2015) was in preparation early in our own work, and are very grateful to Anne Estrade and José León not only for sending us a preprint of the final paper, but also for sharing in-progress versions long before the final preprint was ready. Doing so not only saved duplication of effort, but made our lives much easier.

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2. Gaussian random fields and chaos expansions

Before we set up our results in a formal fashion, we need some preliminaries. In particular, we require a collection of regularity conditions on our random fields which will make the Euler integral well-defined, amenable to analysis, and which are sufficient for our CLT to hold.

In addition, and this will take up most of the section, we need to set up a number of results related to chaos expansions. These will be used in the remainder of the paper to express the Euler integral in this form and then prove our CLT via a general CLT of Nourdin and Peccati (2012) for chaos expansions.

For general preliminaries on random fields and their connection to Morse theory we shall use the often complementary books by Adler and Taylor (2007) and Azaïs and Wschebor (2009), while for a good treatment of the Wiener chaos we rely on Nualart (2006). Results below that we refer to as “standard”, “well known”, or for which we fail to offer even these descriptions, can be found in one of these references.

2.1. Tame Gaussian fields

For the remainder of this paper, X will denote a real valued, mean zero, unit variance, Gaussian random field on \mathbb{R}^n , $n \geq 1$. We denote its covariance (and correlation) function by $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$. For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ we denote its gradient by ∇f , writing this and other vectors as row vectors, and its Hessian by $\nabla^2 f$. We shall occasionally treat $\nabla^2 f$ as a vector rather than a matrix, in which case, because of symmetry, it will have $n(n+1)/2$ elements. It should be clear from the context whether we are using the matrix or vector interpretations. Generic constants, which may change from line to line, are denoted by C .

We write $\text{Cov}(Y)$ for the covariance matrix of a random vector Y , and the ubiquitous symbol $|\cdot|$ to denote all of modulus (of a real number), length (of a vector) and determinant (of a matrix). Again, usage should be clear from the context.

The regularity conditions we shall require on X are summarised in the following definition.

Definition 2.1. *Let $X \triangleq \{X(t), t \in \mathbb{R}^n\}$ be as above. Then we call X tame if the following conditions all hold.*

- (i) *At each $t \in \mathbb{R}^n$, the joint distribution of the vector $\langle X(t), \nabla X(t), \nabla^2 X(t) \rangle$ is non-degenerate.*
- (ii) *The covariance function, ρ , of X is four times differentiable, and for some $\alpha > 0$, and t small enough, each of its four-order derivatives satisfies*

$$\left| \rho^{(4)}(0) - \rho^{(4)}(t) \right| \leq \frac{C}{(-\ln |t|)^{1+\alpha}}. \quad (2.1)$$

(iii) Set

$$\psi(t) \triangleq \sup_{0 \leq m \leq 4} \left| \frac{\partial^m \rho}{\partial t_{i_1} \dots \partial t_{i_m}}(t) \right|. \quad (2.2)$$

Then $\psi \in L^1(\mathbb{R}^n)$, and $\psi(t) \rightarrow 0$ as $|t| \rightarrow \infty$. (iv)

(iv) Let $N_v(\nabla X, M)$ be the number of points, $t \in M$, for which $\nabla X(t) = v$. Then, for any $v \in \mathbb{R}^n$,

$$\mathbb{E} \left[(N_v(\nabla X, M))^3 \right] < \infty.$$

There are a number of immediate, standard, consequences to tameness for a Gaussian random field. In particular, (ii) ensures that the trajectories of X are almost surely (henceforth a.s.) in $C^2(\mathbb{R}^n)$, and, via the exponential integrability of the suprema of Gaussian processes (assured by the Borel-Tsirelson-Ibragimov-Sudakov inequality) that

$$\mathbb{E} \left[\left| \sup_{t \in M} X_t \right|^k \right] < \infty,$$

for any compact domain $M \subset \mathbb{R}^n$, and any $k \geq 1$.

Condition (i) ensures that the realisations of X are a.s. Morse functions. We shall prove later that (iii) ensures the decay of correlation necessary for a CLT to hold. Condition (iv) can be directly verified for specific covariance functions using standard integral expressions for the factorial moments of $N_v(\nabla X, M)$, as in Adler and Taylor (2007)[Theorem 11.5.1]. Alternatively, following Belyaev (1966), this condition can be substituted by requiring non-degeneracy and smoothness for higher order derivatives of the field. (See also a second moment calculation in an isotropic setting in Estrade and León (2015)[Proposition 1.1].)

2.2. Correlation structure of stationary, tame, Gaussian fields

In what follows, we shall often need details about the distribution of the random vector

$$\vec{X} \triangleq \langle (\nabla X)_1, \dots, (\nabla X)_n, X, (\nabla^2 X)_{1,1}, (\nabla^2 X)_{1,2}, \dots, (\nabla^2 X)_{n,n} \rangle,$$

of length $N_n \triangleq 1 + n + n(n+1)/2$, where

$$(\nabla X)_i = \frac{\partial X}{\partial t_i}, \quad \text{and} \quad (\nabla^2 X)_{i,j} = \frac{\partial^2 X}{\partial t_i \partial t_j}, \quad i, j = 1, \dots, n,$$

are the first and second derivatives of X . Since X is tame, all of these derivatives exist, and all are Gaussian. Furthermore, by stationarity, the distribution of \vec{X}_s is independent of s , and the elements of the covariance matrix are given by derivatives of the covariance

function ρ at the origin. It is then well established that the covariance matrix Λ of \vec{X} factorizes as

$$\Lambda = \begin{pmatrix} \Lambda_{(1)} & 0 \\ 0 & \Lambda_{(2)} \end{pmatrix}, \quad (2.3)$$

where $\Lambda_{(1)}$ is the covariance matrix of ∇X and $\Lambda_{(2)}$ is the covariance matrix of $\langle X, \nabla^2 X \rangle$. Now let $\Lambda^{(1/2)}$ be a square root of Λ , and define the random field Y by

$$Y(s) \triangleq \Lambda^{-(1/2)} \vec{X}(s). \quad (2.4)$$

The representation (2.3) induces a similar factorization on $\Lambda^{(1/2)}$, $\Lambda^{-(1/2)}$, and Y . It is important to note that Y is a vector valued random field. Furthermore, since, for each s , $Y(s)$ is a vector of independent, standard normal variables, we shall call Y the *decorrelated version* of \vec{X} . However, note that despite the independence of the elements of $Y(s)$ for each s , the vectors $Y(s)$ and $Y(t)$ are not independent for $s \neq t$.

For later needs, note that if we define the covariance matrix K by

$$K(t) \equiv \text{Cov}(Y)(t) \triangleq \mathbb{E}[Y_0' Y_t], \quad (2.5)$$

then it is easy to check that its entries $\{(K(t))_{ij}\}_{i,j=1}^{N_n}$, are bounded by

$$\begin{aligned} |(K(t))_{jk}| &\leq \|\Lambda^{-(1/2)}\| \cdot \|\mathbb{E}[\vec{X}_0' \vec{X}_t]\| \cdot \|[\Lambda^{-(1/2)}]'\| \\ &\leq n^2 \|\Lambda^{-(1/2)}\|^2 \cdot \psi(t) \\ &\leq C\psi(t). \end{aligned} \quad (2.6)$$

Here ψ is given by (2.2) and C is a constant dependent only on the derivatives of ρ at the origin.

Throughout this work, \vec{Y} denotes the decorrelated version of \vec{X} . Before concluding this section and while the definition of \vec{Y} is still fresh in our memory, we introduce a new notation, and using this new notation state some of the relations between \vec{Y} and X that we will need later on:

- For an arbitrary vector \vec{u} of dimension d and a set of indexes $\mathcal{I} = \{i_j\}_{j=1}^k$ where $i_j \in \{1, \dots, n\}$ and $k \leq d$, we define the vector $\vec{V}_{\mathcal{I}}(\vec{u})$ by

$$\vec{V}_{\mathcal{I}}(\vec{u}) \triangleq (u_{i_1}, u_{i_2}, \dots, u_{i_k}). \quad (2.7)$$

In particular, with $\mathcal{I} = \{N_n - n\}$ we have

$$X = \vec{V}_{\mathcal{I}} \left(\Lambda_{(2)}^{(1/2)} \vec{Y}_{(2)} \right),$$

and with $\mathcal{I} = \{m, l\}$

$$((\nabla X)_m, (\nabla X)_l) = \vec{V}_{\mathcal{I}} \left(\Lambda_{(1)}^{(1/2)} \vec{Y}_{(1)} \right).$$

- For an arbitrary vector \vec{u} and a set of indexes $\mathcal{I} = \{i_j\}_{j=1}^k$, we implicitly define a symmetric matrix $\mathbf{M}_{\mathcal{I}}(\vec{u})$ constructed from the elements of \vec{u} so that, when $\vec{u} = \left(\Lambda_{(2)}^{(1/2)} \vec{Y}_{(2)}\right)$ and $\mathcal{I} = \{i_j, \dots, i_k\}$, we have

$$\mathbf{M}_{\mathcal{I}}\left(\Lambda_{(2)}^{(1/2)} \vec{Y}_{(2)}\right) \triangleq \begin{pmatrix} (\nabla^2 X)_{i_1 i_1} & (\nabla^2 X)_{i_1 i_2} & \cdots & (\nabla^2 X)_{i_1 i_k} \\ (\nabla^2 X)_{i_2 i_1} & (\nabla^2 X)_{i_2 i_2} & \cdots & (\nabla^2 X)_{i_2 i_k} \\ \vdots & \vdots & \cdots & \vdots \\ (\nabla^2 X)_{i_k i_1} & (\nabla^2 X)_{i_k i_2} & \cdots & (\nabla^2 X)_{i_k i_k} \end{pmatrix}. \quad (2.8)$$

In particular, if $\mathcal{I} = \{1, \dots, n\}$, then

$$\nabla^2 X = \mathbf{M}_{\mathcal{I}}\left(\Lambda_{(2)}^{(1/2)} \vec{Y}_{(2)}\right). \quad (2.9)$$

2.3. The spectral distribution of Y

Since X , and so its decorrelated related version Y , are stationary, both have spectral representations. While all properties of the spectral representation of X follow from the classical theory (e.g. Yaglom (1962)) we need to work a little to set up an appropriate representation for the vector valued field Y . In particular, we shall do this in the language of isonormal processes, which provides the necessary structure for later proofs.

We start by noting that since X is tame, the function ψ is integrable, and so by (2.6) the same is true of the covariance K of Y . Consequently, by standard spectral theory, Y has a matrix valued spectral density function f for which

$$(K(\tau))_{jk} = \mathbb{E}[Y_j(0)Y_k(\tau)] = \int_{\mathbb{R}^n} e^{i\langle \tau, \lambda \rangle} f_{jk}(\lambda) d\lambda, \quad \tau \in \mathbb{R}^n. \quad (2.10)$$

It is not too difficult to express the f_{ij} in terms of the spectral density of X (which is, essentially, the only ‘free parameter’ in the entire setup) but, fortunately, their explicit form will not be important in what follows. What is important, however, and follows for the non-degeneracy condition (i) of tameness, is that, for all $\lambda \in \mathbb{R}^n$, $(f_{jk}(\lambda))_{j,k=1}^{N_n}$ is a symmetric, positive semi-definite matrix, and so has a symmetric square root $(b_{jk}(\lambda))_{j,k=1}^{N_n}$. Consequently, we also have that

$$f_{jk}(\lambda) = \sum_l b_{jl}(\lambda) b_{lk}(\lambda). \quad (2.11)$$

2.4. Representing Y via the isonormal process

Retaining the notation of the previous subsection, we start with a separable Hilbert space \mathfrak{H} of Hermitian functions

$$\mathfrak{H} \triangleq \left\{ h(j, \vec{\lambda}) : \{1, \dots, N_n\} \times \mathbb{R}^n \rightarrow \mathbb{C} \mid \overline{h(j, \vec{\lambda})} = h(j, -\vec{\lambda}), \|h\|_{\mathfrak{H}}^2 < \infty \right\}, \quad (2.12)$$

with the inner product

$$\langle h, g \rangle_{\mathfrak{H}} \triangleq \sum_{j_1=1}^{N_n} \sum_{j_2=1}^{N_n} \int_{\mathbb{R}^n} h(j_1, \vec{\lambda}) f_{j_1, j_2}(\vec{\lambda}) \overline{g(j_2, \vec{\lambda})} d\lambda. \quad (2.13)$$

Next, we let $W^{(j)}$, $j = 1, \dots, N_n$ a sequence of independent, real-valued, Gaussian white noises on \mathbb{R}^n , and use them to define a random process over $h \in \mathfrak{H}$ by

$$W(h) \triangleq \sum_{j=1}^{N_n} \sum_{k=1}^{N_n} \int_{\mathbb{R}^n} h(j, \vec{\lambda}) b_{jk}(\vec{\lambda}) W^{(k)}(d\lambda), \quad (2.14)$$

the integrals here all being standard stochastic integrals.

By construction $W(h)$ is centered Gaussian and $\mathbb{E}[W(h)W(g)] = \langle f, g \rangle_{\mathfrak{H}}$. Moreover, since the functions h are Hermitian the resulting random process is real valued. It is known as the *isonormal Gaussian process* on \mathfrak{H} .

Finally, we want to relate Y to W , as promised. To this end define a new family of functions, $\varphi_{t,k}(j, \lambda)$, $k = 1, \dots, N_n$, $t \in \mathbb{R}^n$, in \mathfrak{H} via

$$\varphi_{t,m} \equiv \varphi_{t,m}(j, \vec{\lambda}) \triangleq e^{i\langle \vec{t}, \vec{\lambda} \rangle} \delta_{j,m} \quad j, m = 1, \dots, N_n, \quad \vec{t}, \vec{\lambda} \in \mathbb{R}^n. \quad (2.15)$$

It is straightforward to check that

$$\mathbb{E}[W(\varphi_{\vec{t}_1, l})W(\varphi_{\vec{t}_2, m})] = \langle \varphi_{\vec{t}_1, l}, \varphi_{\vec{t}_2, m} \rangle_{\mathfrak{H}} = \mathbb{E}[Y_m(0)Y_l(t_1 - t_2)].$$

An immediate consequence of this is that the vector valued random field Y has the following particularly useful L_2 representation in terms of the isonormal process and the family $\varphi_{s,k}$:

$$Y_l(s) \stackrel{L_2}{=} W(\varphi_{s,l}), \quad l = 1, \dots, N_n, \quad s \in \mathbb{R}^n. \quad (2.16)$$

Note that it also follows from these calculations that

$$\|\varphi_{s,k}\|_{\mathfrak{H}} = \mathbb{E}[Y_k(0)Y_k(0)] = 1, \quad (2.17)$$

and

$$\langle \varphi_{s,k}, \varphi_{s,m} \rangle_{\mathfrak{H}} = \mathbb{E}[W(\varphi_{s,l})W(\varphi_{s,m})] = \delta_{k,m}, \quad (2.18)$$

where $\delta_{k,m}$ is the Kronecker delta.

2.5. Operations on $f \in \mathfrak{H}$

We now describe the basic operations on \mathfrak{H} that we will need later. Let $\{\vec{e}_j\}_{j \geq 1}$ be an orthonormal family of functions in \mathfrak{H} , and write $\mathbb{M} \triangleq \{1, \dots, N_n\} \times \mathbb{R}^n$. Then, $\vec{e}_j(\lambda) : \mathbb{M} \rightarrow \mathbb{C}, j \geq 1$, and we define the following operations:

- Tensor product, $\vec{e}_j \otimes \vec{e}_k : \mathbb{M} \times \mathbb{M} \rightarrow \mathbb{C}$:

$$[\vec{e}_j \otimes \vec{e}_k](\lambda_1, \lambda_2) \triangleq \vec{e}_j(\lambda_1) \vec{e}_k(\lambda_2), \quad \lambda_1, \lambda_2 \in \mathbb{M}, \quad (2.19)$$

where $\vec{e}_j \otimes \vec{e}_k$ belongs to the Hilbert space $\mathfrak{H}^{\otimes 2}$, with the inner product induced, component-wise, by the inner product in \mathfrak{H} :

$$\langle \vec{e}_j \otimes \vec{e}_k, \vec{e}_l \otimes \vec{e}_m \rangle_{\mathfrak{H}^{\otimes 2}} \triangleq \langle \vec{e}_j, \vec{e}_l \rangle_{\mathfrak{H}} \langle \vec{e}_k, \vec{e}_m \rangle_{\mathfrak{H}}. \quad (2.20)$$

- In a similar fashion, we define the m -fold tensor product of \vec{e}_j with itself:

$$\vec{e}_j^{\otimes m} \triangleq \underbrace{\vec{e}_j \otimes \dots \otimes \vec{e}_j}_{m \text{ times}}, \quad (2.21)$$

where $\vec{e}_j^{\otimes m}$ belongs to the Hilbert space $\mathfrak{H}^{\otimes m}$, with the inner product defined component-wise by the inner product in \mathfrak{H} . Likewise, for $\vec{e}_j^{\otimes q} \in \mathfrak{H}^{\otimes q}$, $\vec{e}_k^{\otimes p} \in \mathfrak{H}^{\otimes p}$, the tensor product of higher order in $\mathfrak{H}^{\otimes q+p}$ is

$$\vec{e}_j^{\otimes q} \otimes \vec{e}_k^{\otimes p} \triangleq \underbrace{\vec{e}_j \otimes \dots \otimes \vec{e}_j}_{q \text{ times}} \otimes \underbrace{\vec{e}_k \otimes \dots \otimes \vec{e}_k}_{p \text{ times}}. \quad (2.22)$$

- Take $0 \leq r \leq p \leq m$. The r -contraction, $(\vec{e}_{j_1} \otimes \dots \otimes \vec{e}_{j_p}) \otimes_r (\vec{e}_{k_1} \otimes \dots \otimes \vec{e}_{k_m})$, is in $\mathfrak{H}^{\otimes p+m-2r}$ and, for $r = 0$, is defined as

$$(\vec{e}_{j_1} \otimes \dots \otimes \vec{e}_{j_p}) \otimes_0 (\vec{e}_{k_1} \otimes \dots \otimes \vec{e}_{k_m}) \triangleq (\vec{e}_{j_1} \otimes \dots \otimes \vec{e}_{j_p} \otimes \vec{e}_{k_1} \otimes \dots \otimes \vec{e}_{k_m}), \quad (2.23)$$

while, for $1 \leq r \leq p \leq m$,

$$\begin{aligned} (\vec{e}_{j_1} \otimes \dots \otimes \vec{e}_{j_p}) \otimes_r (\vec{e}_{k_1} \otimes \dots \otimes \vec{e}_{k_m}) &\triangleq \\ &\triangleq \left[\prod_{l=1}^r \langle \vec{e}_{j_l}, \vec{e}_{k_l} \rangle_{\mathfrak{H}} \right] (\vec{e}_{j_{r+1}} \otimes \dots \otimes \vec{e}_{j_p} \otimes \vec{e}_{k_{r+1}} \otimes \dots \otimes \vec{e}_{k_m}). \end{aligned} \quad (2.24)$$

For $r = p = m$ we have

$$(\vec{e}_{j_1} \otimes \dots \otimes \vec{e}_{j_p}) \otimes_p (\vec{e}_{k_1} \otimes \dots \otimes \vec{e}_{k_m}) = \left[\prod_{l=1}^r \langle \vec{e}_{j_l}, \vec{e}_{k_l} \rangle_{\mathfrak{H}} \right]. \quad (2.25)$$

- Symmetrization of $f \in \mathfrak{H}^{\otimes q}$

$$\tilde{f} \equiv \text{symm}(f) \triangleq \frac{1}{q!} \sum_{\sigma_q} f(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(q)}), \quad (2.26)$$

where σ_q is all the permutations over the indexes $\{1, \dots, q\}$. We write $\mathfrak{H}^{\odot q} \subset \mathfrak{H}^{\otimes q}$ for the space of all symmetric $f \in \mathfrak{H}^{\otimes q}$.

2.6. Wiener chaos expansion

Take $W(h)$ to be an isonormal Gaussian process on separable Hilbert space \mathfrak{H} . Write $\mathcal{G} \triangleq \sigma(W(h))$, for the σ -field generated by the random variables $\{W(h), h \in \mathfrak{H}\}$ and $L^2(\mathcal{G}, \mathbb{R})$ for the space of all square integrable mappings from $(\Omega, \mathcal{G}, \mathbb{P})$ to \mathbb{R} .

We make use of $\{H_n\}_{n \in \mathbb{N}}$, the probabilistic Hermite polynomials, defined by

$$\exp(tx - \frac{t^2}{2}) = \sum_{m=0}^{\infty} H_m(x) \frac{t^m}{m!}, \quad x \in \mathbb{R}^n, \quad (2.27)$$

and define

$$H_{\vec{a}}(x) \triangleq \prod_{j=1}^{\infty} H_{a_j}(x_j), \quad x_j \in \mathbb{R}^n, \quad a_j \in \mathbb{N}, \quad j = 1, \dots, \infty. \quad (2.28)$$

Here, the sequence $\vec{a} = \{a_1, a_2, \dots\}$ is such that only a finite number of elements differs from zero. One can make use of $\{H_{\vec{a}}(x)\}$ to construct an orthonormal basis for $L^2(\mathcal{G}, \mathbb{R})$. The basis is given by the random variables

$$\Phi_{\vec{a}} \triangleq \sqrt{\vec{a}!} \prod_{j=1}^{\infty} H_{a_j}(W(\vec{e}_j)), \quad \vec{a}! \triangleq \prod_{i=0}^{\infty} a_i! , \quad (2.29)$$

with $W(h), h \in \mathfrak{H}$ the isonormal process as defined in Section 2.4 and $\{\vec{e}_j\}_{j \geq 1}$ the orthonormal basis of \mathfrak{H} .

Moreover, the space $L^2(\mathcal{G}, \mathbb{R})$ may be represented as the decomposition of a countable set of closed orthogonal subspaces $\{\mathcal{H}_m\}_{m \in \mathbb{N}}$,

$$L^2(\mathcal{G}, \mathbb{R}) = \oplus_{m=0}^{\infty} \mathcal{H}_m \quad (2.30)$$

such that, for each m , $\{\Phi_{\vec{a}}\}_{|\vec{a}|=m}$ is a complete orthogonal system in $\mathcal{H}_m \subset L^2(\mathcal{G}, \mathbb{R})$. Thus, given $F \in L^2(\mathcal{G}, \mathbb{R})$ there is a unique decomposition

$$F = \sum_{m=0}^{\infty} \sum_{|\vec{a}|=m} \mathbb{E}[\Phi_{\vec{a}} F] \Phi_{\vec{a}}, \quad |\vec{a}| \triangleq \sum_{i=0}^{\infty} a_i , \quad (2.31)$$

with $\sum_{|\vec{a}|=m} \mathbb{E}[\Phi_{\vec{a}} F] \Phi_{\vec{a}} \in \mathcal{H}_m$, orthogonal for different values of m .

We are now in a position to set up the Wiener chaos expansion. To this end define $I_q : \mathfrak{H}^{\odot q} \rightarrow L^2(\mathcal{G}, \mathbb{R})$ by

$$I_m(\vec{e}_{j_1} \otimes \cdots \otimes \vec{e}_{j_q}) = \sqrt{a!} \prod_{i=1}^q H_{a_i}(W(\vec{e}_{j_i})), \quad (2.32)$$

where $a_i = \#\{k : j_k = i\}$. Using $\{I_q(\cdot)\}$, one can rewrite (2.31) as

$$F = \sum_{q=0}^{\infty} I_q(f_q), \quad f_q \in \mathfrak{H}^{\odot q}. \quad (2.33)$$

This representation of the random functional F via the set of kernels $\{f_q\}$ through the family of linear operations, I_q , is called the *Wiener chaos decomposition* of F , and the operator I_q is called the *multiple Wiener integral* of order q .

If the underlying Hilbert space is a Polish space of the form $L^2(A, \mathcal{A}, \mu)$, then I_q can be identified with multiple stochastic integrals. More specifically, take $\{A_i\} \in \mathcal{A}$ disjoint sets, and define

$$u_q = \text{symm} \left[\sum a_{i_1, \dots, i_q} \mathbb{1}_{A_1} \otimes \cdots \otimes \mathbb{1}_{A_q} \right]. \quad (2.34)$$

The functions u_q are dense in $\text{symm}[(L^2(A, \mathcal{A}, \mu))^{\otimes q}]$, and the integral is constructed first on the functions u_q by defining

$$I(u_q) = \sum a_{i_1, \dots, i_q} W(A_1) \cdots W(A_q), \quad (2.35)$$

and then extending to a linear continuous operator on all of $(L^2(A, \mathcal{A}, \mu))^{\odot q}$. We write

$$I(f_q) = \int_A \cdots \int_A f_q(t_1, \dots, t_q) W(d\mu) \cdots W(d\mu) \quad t_i \in A, \quad (2.36)$$

with $W(d\mu)$ a Gaussian μ -noise.

Below we list some of the basic properties of I_q that will be of particular interest to us. For more details, see (Nualart, 2006, Ch. 1). For an in depth treatment related to the approach above, see Nourdin and Peccati (2012).

- $f \in \mathfrak{H}^{\otimes m}$, $I_m(f) = I_m(\tilde{f})$.
- For any set of $h_i \in \mathfrak{H}$, such that $\|h_i\|_{\mathfrak{H}} = 1, \forall i$, we have

$$\prod_{i=1}^m H_{a_i}(W(h_i)) = I_{|\vec{a}|}(\text{symm}(h_1^{\otimes a_1} \otimes \cdots \otimes h_m^{\otimes a_m})). \quad (2.37)$$

- $\mathbb{E}I_p(f)I_q(g) = 0$ when $p \neq q$ and $\mathbb{E}I_p(f)I_q(g) = p!\langle \tilde{f}, \tilde{g} \rangle_{\mathfrak{H}^{\otimes p}}$ when $p = q$.

Other properties of the multiple Wiener integral will be recalled when needed.

It is in this language of Wiener chaos that we seek to represent the Euler integral, and then make use of the following result.

Theorem 2.1 (Theorem 6.3.1, Nourdin and Peccati (2012)). *Let $(F_m)_{m \geq 1}$ be a sequence in $L^2(\Omega, \mathcal{G}, P)$ such that $\mathbb{E}[F_m] = 0 \ \forall m$. Suppose that the chaos expansion of F is given by $F_m = \sum_{q=1}^{\infty} I_q(f_q^m)$, and suppose in addition that*

- (a) $q! \|f_q^m\|_{\mathfrak{H}^{\otimes q}}^2 \rightarrow \sigma_q^2$ as $m \rightarrow \infty$, for some $\sigma_q^2 \geq 0$.
- (b) $\sigma_F^2 \triangleq \sum_{q=1}^{\infty} \sigma_q^2 < \infty$.
- (c) $\forall q \geq 2$ and $r = 1, \dots, q-1$, $\|f_q^m \otimes_r f_q^m\|_{\mathfrak{H}^{\otimes 2(q-r)}}^2 \rightarrow 0$ as $m \rightarrow \infty$.
- (d) $\lim_{Q \rightarrow \infty} \sup_{m \geq 1} \sum_{q=Q+1}^{\infty} q! \|f_q^m\|_{\mathfrak{H}^{\otimes q}}^2 \rightarrow 0$.

Then, $F_m \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_F^2)$ as $m \rightarrow \infty$.

2.7. A Rice type formula and a Morse theoretical representation of the Euler integral

Although up until now we have approached the Euler integral via integration theory, for the proofs to follow we require a slightly different approach, via stratified Morse theory. This theory links the topology of sets to the study of critical points defined on them. In particular, the Euler characteristic of excursion sets of the form

$$A(M, u) \triangleq \{t \in M : f(t) \geq u\},$$

is easily computed via properties of the critical points of f .

To see how this works in our setting, we return to Section 1.1, with $M = T_n \triangleq [0, m]^n$, and, noting that $\chi(T_n) = 1$, obtain

$$\int_{T_n} f[d\chi] = \int_{u=0}^{\infty} [1 - \chi(T_n \cap (f \leq u)) - \chi(T_n \cap (f \leq -u))] du. \quad (2.38)$$

We introduce $\mu(\vec{s})$, the Morse index of a critical point \vec{s} of f . (i.e. $\nabla f(\vec{s}) = 0$ and the Hessian $\nabla^2 f(\vec{s})$ has $\mu(\vec{s})$ negative eigenvalues.) We write T_n° for the interior of the cube, then proceed to decompose the boundary of T_n into open faces each of which is an open cube of dimension less than n . (The vertexes are considered to be zero dimensional closed cubes). To save on notation we denote any such face by J , and write $\{J\}$ to denote the collection of all such faces. When quantities are evaluated with respect to a particular face, this will be denoted by an appropriate subscript. With the above notation, a careful

application of Theorem 9.3.5 of Adler and Taylor (2007) (see also Bobrowski and Borman (2012)) yields

$$\begin{aligned} \int_{T_n} f[\chi] = & \sum_{\{\vec{s} \in T_n^\circ : \nabla f(\vec{s}) = \vec{0}\}} (-1)^{\mu(\vec{s})} f(\vec{s}) + \\ & + \sum_{J \in \{J\} \setminus T_n^\circ} \left(\sum_{\{\vec{s} \in J : \nabla f|_J(\vec{s}) = \vec{0}\}} (-1)^{\dim J - \mu_{f|J}(\vec{s})} f(\vec{s}) \mathbb{1}_{\{\langle \nabla f(\vec{s}), \vec{\eta}_J \rangle \geq 0\}} \right). \end{aligned} \quad (2.39)$$

In the above, $\{\vec{\eta}_J\}$ are constant vectors attached to every face of the cube T_n (the details can be found in Adler and Taylor (2007)). We identify

$$\sum_{\{\vec{s} \in T_n^\circ : \nabla f(\vec{s}) = \vec{0}\}} (-1)^{\mu(\vec{s})} f(\vec{s}) \quad (2.40)$$

as the contribution of the internal critical points to the Euler integral, and

$$\sum_{J \in \{J\} \setminus T_n^\circ} \left(\sum_{\{\vec{s} \in J : \nabla f|_J(\vec{s}) = \vec{0}\}} (-1)^{\dim J - \mu_{f|J}(\vec{s})} f(\vec{s}) \mathbb{1}_{\{\langle \nabla f(\vec{s}), \vec{\eta}_J \rangle \geq 0\}} \right) \quad (2.41)$$

as the contribution of the critical points on the boundary.

From a critical point representation of this kind, one can develop an integral representation of Rice type, and it is this that will be at the core of all the proofs to follow.

Lemma 2.2. *Let f be a Morse function. Then*

$$\begin{aligned} \int_{T_n} f[d\chi] = & \lim_{\sigma \rightarrow 0} \int_{T_n} \phi_{\sigma^2 \mathbf{I}_{n \times n}}(\nabla f|_{T_n^\circ}(\vec{s})) \det(\nabla^2 f|_{T_n^\circ}(\vec{s})) f|_{T_n^\circ}(\vec{s}) d\vec{s} + \\ & + \sum_{J \in \{J\} \setminus T_n^\circ} (-1)^{\dim J} \lim_{\sigma \rightarrow 0} \int_J \phi_{\sigma^2 \mathbf{I}_{\dim J \times \dim J}}(\nabla f|_J(\vec{s})) \det(\nabla^2 f|_J(\vec{s})) f|_J(\vec{s}) \mathbb{1}_{\{\langle \nabla f(\vec{s}), \vec{\eta}_J \rangle \geq 0\}} d\vec{s}, \end{aligned} \quad (2.42)$$

where $\phi_{\sigma^2 \mathbf{I}_{\dim J \times \dim J}}(\vec{s})$ is a $(\dim J)$ -dimensional centered Gaussian kernel with the covariance matrix $\sigma^2 \mathbf{I}_{\dim J \times \dim J}$.

Proof. Standard techniques for the construction of Rice type integral formulae, along with the fact that the $\sigma \rightarrow 0$ limit of $\phi_{\sigma^2 \mathbf{I}}$ is a Dirac delta, establish that, for any $J \in \{J\}$,

$$\begin{aligned} \lim_{\sigma \rightarrow 0} \int_{T_n} \phi_{\sigma^2 \mathbf{I}}(\nabla f(\vec{s})) |\det(\nabla^2 f(\vec{s}))| f(\vec{s}) d\vec{s} = & \int_{T_n} (\delta \circ \nabla f)(\vec{s}) f(\vec{s}) d\vec{s} \\ = & \sum_{\{\vec{s} \in T_n^\circ : \nabla f(\vec{s}) = \vec{0}\}} f(\vec{s}), \end{aligned} \quad (2.43)$$

and

$$\begin{aligned}
& \lim_{\sigma \rightarrow 0} \int_J \phi_{\sigma^2} \mathbf{I}_{\dim J \times \dim J} (\nabla f|_J(\vec{s})) \left| \det(\nabla^2 f|_J(\vec{s})) \right| f|_J(\vec{s}) \mathbb{1}_{\{\langle \nabla f(\vec{s}), \vec{\eta}_J \rangle \geq 0\}} d\vec{s} \\
&= \int_J (\delta \circ \nabla f|_J)(\vec{s}) f|_J(\vec{s}) \mathbb{1}_{\{\langle \nabla f(\vec{s}), \vec{\eta}_J \rangle \geq 0\}} d\vec{s} \\
&= \sum_{\{\vec{s} \in J: \nabla f|_J(\vec{s}) = \vec{0}\}} f(\vec{s}) \mathbb{1}_{\{\langle \nabla f(\vec{s}), \vec{\eta}_J \rangle \geq 0\}}. \tag{2.44}
\end{aligned}$$

Using the fact that the determinant of a matrix equals the product of its eigenvalues, it follows that $\text{sign}\{\det(\nabla^2 f|_J)(\vec{s})\} = (-1)^{\mu|J(\vec{s})}$. Thus, we can drop the absolute value in (2.43) and (2.44), and, applying (2.39), complete the proof. \square

We now have all that we need to formulate, and to prove, the main result of this paper.

3. A CLT for the Euler integral

Theorem 3.1. *Let $X \triangleq \{X(\vec{s}) | \vec{s} \in \mathbb{R}^n\}$ be a tame Gaussian field, as in Definition 2.1. Then, the (upper) Euler integral*

$$\Psi_{[0,m]^n}[X] \triangleq \int_{[0,m]^n} X(\vec{s}) [d\chi]$$

satisfies the central limit theorem

$$\frac{\Psi_{[0,m]^n}[X] - \mathbb{E}[\Psi_{[0,m]^n}[X]]}{m^{n/2}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_\Psi^2), \text{ as } m \rightarrow \infty.$$

where $\sigma_\Psi^2 > 0$ is defined by (3.36) below.

Note that an expression for the mean value of Euler integral, $\mathbb{E}[\Psi_{[0,m]^n}[X]]$, was derived in Bobrowski and Borman (2012), and is also discussed in Section 4 below from the point of view of chaos expansions.

Before starting the proof of Theorem 3.1, note that while expressions like (2.39) and (2.42) relate to the full Euler integral, only the first sum in (2.39) and the first integral in (2.42), which relate to contributions from the interior of T_n , are relevant for the CLT. The reason for this lies in the normalisation of $m^{-n/2}$, which applies equally to all terms. It follows from the calculations of this subsection that all non-interior terms, when normalised by $m^{-n/2}$, converge in probability to zero, and so not affect the limiting distribution. We leave the (simple) details of this to the reader, and so in dealing with the CLT henceforth concentrate only on the interior terms.

We start with a sequence of lemmas, which will ultimately be combined to provide a full proof of Theorem 3.1 in the following subsection.

3.1. Four supporting lemmas

Lemma 3.1. *Let X be a tame Gaussian random field. Let*

$$F_{(0,m)^n}[X] = \sum_{\{\vec{s} \in T_n^\circ : \nabla X(\vec{s}) = \vec{0}\}} (-1)^{\mu(\vec{s})} X(\vec{s}), \quad (3.1)$$

and

$$F_{(0,m)^n}^\sigma[X] = \int_{(0,m)^n} \phi_{\sigma^2 \mathbf{I}}(\nabla X(\vec{s})) \det(\nabla^2 X(\vec{s})) X(\vec{s}) d\vec{s}. \quad (3.2)$$

Then,

$$F_{(0,m)^n}^\sigma[X] \xrightarrow{L_2} F_{(0,m)^n}[X] \quad \text{as } \sigma \rightarrow 0. \quad (3.3)$$

Proof. We deduce the L^2 convergence from the following two facts:

- (a) $F_{(0,m)^n}^\sigma[X] \xrightarrow{a.s.} F_{(0,m)^n}[X]$ as $\sigma \rightarrow 0$,
- (b) There exists $\varepsilon > 0$ so that $\sup_\sigma \mathbb{E}[F_{(0,m)^n}^\sigma[X]]^{2+\varepsilon} < \infty$.

To prove (a), note that the trajectory of X is almost surely Morse and the result then follows from Lemma 2.2. To show (b), we write an upper bound for $\mathbb{E}[F_{(0,m)^n}^\sigma[X]]^{2+\varepsilon}$ independent of σ . To this end, using Federer's coarea formula (cf. Azaïs and Wschebor (2009), Proposition 6.1, for a version couched in our terminology) we have

$$\int_{(0,m)^n} \phi_{\sigma^2 \mathbf{I}}(\nabla X(\vec{s})) |\det(\nabla^2 X(\vec{s}))| d\vec{s} = \int_{\mathbb{R}^n} \phi_{\sigma^2 \mathbf{I}}(\vec{u}) N_{\vec{u}}(\nabla X, (0, m)^n) d\vec{u}. \quad (3.4)$$

Write $X_{\text{sup}} \triangleq \sup_{\vec{s} \in (0,m)^n} |X(\vec{s})|$. Then, applying (3.4), we obtain

$$\int_{(0,m)^n} \phi_{\sigma^2 \mathbf{I}}(\nabla X(\vec{s})) \det(\nabla^2 X(\vec{s})) X(\vec{s}) d\vec{s} \leq |(X_{\text{sup}})| \left| \int_{\mathbb{R}^n} \phi_{\sigma^2 \mathbf{I}}(\vec{u}) N_{\vec{u}}(\nabla X, (0, m)^n) d\vec{u} \right|.$$

Using the fact that $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ for $a, b > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} & \left| (X_{\text{sup}}) \int_{\mathbb{R}^n} \phi_{\sigma^2 \mathbf{I}}(\vec{u}) N_{\vec{u}}(\nabla X, (0, m)^n) d\vec{u} \right|^{(2+\varepsilon)} \\ & \leq \frac{1}{p} \left| (X_{\text{sup}})^{(p(2+\varepsilon))} \right| + \frac{1}{q} \left| \int_{\mathbb{R}^n} \phi_{\sigma^2 \mathbf{I}}(\vec{u}) N_{\vec{u}}(\nabla X, (0, m)^n) d\vec{u} \right|^{q(2+\varepsilon)}. \end{aligned} \quad (3.5)$$

Taking expectations yields

$$\begin{aligned} \mathbb{E} \left[\left| \int_{(0,m)^n} \phi_{\sigma^2 \mathbf{I}}(\nabla X(\vec{s})) \det(\nabla^2 X(\vec{s})) X(\vec{s}) d\vec{s} \right| \right]^{(2+\varepsilon)} \\ \leq \frac{1}{p} \mathbb{E} |(X_{\text{sup}})|^{p(2+\varepsilon)} + \frac{1}{q} \mathbb{E} \left[\int_{\mathbb{R}^n} \phi_{\sigma^2 \mathbf{I}}(\vec{u}) N_{\vec{u}}(\nabla X, (0, m)^n) d\vec{u} \right]^{q(2+\varepsilon)}. \end{aligned} \quad (3.6)$$

$\mathbb{E} |(X_{\text{sup}})|^{p(2+\varepsilon)}$ is finite due to our assumptions of tameness on X , so we focus on the second term in (3.6), viz.

$$\mathbb{E} \left[\int_{\mathbb{R}^n} \phi_{\sigma^2 \mathbf{I}}(\vec{u}) N_{\vec{u}}(\nabla X, (0, m)^n) d\vec{u} \right]^{q(2+\varepsilon)}. \quad (3.7)$$

Jensen's inequality, when applied to the inner integral (and not to the expectation), implies that the above can be bounded by

$$\mathbb{E} \left[\int_{\mathbb{R}^n} \phi_{\sigma^2 \mathbf{I}}(\vec{u}) [N_{\vec{u}}(\nabla X, (0, m)^n)]^{(2+\varepsilon)} d\vec{u} \right].$$

Using Tonelli's theorem, this equals

$$\int_{\mathbb{R}^n} \phi_{\sigma^2 \mathbf{I}}(\vec{u}) \mathbb{E} [N_{\vec{u}}(\nabla X, (0, m)^n)]^{q(2+\varepsilon)} d\vec{u},$$

and, finally (for $q = 1 + \varepsilon$) under the assumptions of a tameness we have

$$\mathbb{E} N_{\vec{u}}(\nabla X, (0, m)^n)^{(2+\varepsilon)} \leq M,$$

and we are done.

For the other faces $J \in \{J\} \setminus T_n^\circ$, of dimension $d \equiv \dim(J) < n$, we have

$$\begin{aligned} \left| \mathbb{E} \left[\int_J \phi_{\sigma^2 \mathbf{I}}(\nabla X(\vec{s})) \det(\nabla^2 X(\vec{s})) X(\vec{s}) \mathbb{1}_{\{\langle \nabla X(\vec{s}), \eta_J \rangle \geq 0\}} d\vec{s} \right] \right|^{(2+\varepsilon)} \\ \leq \mathbb{E} \left[(X_{\text{sup}}) \int_{\mathbb{R}^d} \phi_{\sigma^2 \mathbf{I}}(\vec{u}) N_{\vec{u}}(\nabla X, J) d\vec{u} \right]^{(2+\varepsilon)}, \end{aligned} \quad (3.8)$$

and can then repeat the same argument as above. \square

For the next lemma, which deals with the Wiener chaos decomposition of $F_{(0,m)^n}[X]$, we introduce the notations

$$\pi_n(q) \equiv \{\vec{a} \mid a_1 + \dots + a_{N_n} = q\},$$

and

$$\tilde{H}_{\vec{a}}(Y_s) \equiv \prod_{i=1}^{N_n} H_{a_i}(Y_i(s)).$$

Lemma 3.2. $F_{(0,m)^n}^\sigma[X]$ admits the Wiener chaos expansion

$$F_{(0,m)^n}^\sigma[X] \stackrel{L_2}{=} \sum_{q=1}^{\infty} \sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}}^\sigma \int_{(0,m)^n} \tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) d\vec{s}, \quad (3.9)$$

where

$$d_{\vec{a}}^\sigma = \frac{1}{\vec{a}!} \int_{\mathbb{R}^n} \phi_{\sigma^2 \mathbf{I}_{n \times n}} \left(\mathbf{\Lambda}_{(1)}^{(1/2)} \vec{v} \right) \prod_{i=1}^n H_{a_i}(v_i) \phi(v_i) d\vec{v} \quad (3.10)$$

$$\times \int_{\mathbb{R}^{(N_n-n)}} \det \left(\mathbf{M}_{\mathcal{I}} \left(\mathbf{\Lambda}_{(2)}^{(1/2)} \vec{u} \right) \right) \vec{V}_{\{N_n-n\}} \left(\mathbf{\Lambda}_{(2)}^{(1/2)} \vec{u} \right) \prod_{i=1}^{N_n-n} H_{a_{n+i}}(u_i) \phi(u_i) d\vec{u}. \quad (3.11)$$

Proof. Consider

$$F_{(0,m)^n}^\sigma[X] = \int_{(0,m)^n} \phi_{\sigma^2 \mathbf{I}}(\mathbf{\Lambda}_{(1)}^{(1/2)} \vec{Y}_{(1)}) \det \left(\mathbf{M}_{\mathcal{I}} \left(\mathbf{\Lambda}_{(2)}^{(1/2)} \vec{Y} \right) \right) \vec{V}_{\{N_n-n\}} \left(\mathbf{\Lambda}_{(2)}^{(1/2)} \vec{Y} \right) d\vec{s}.$$

Take $f_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ and $f_2 : \mathbb{R}^{N_n-n} \rightarrow \mathbb{R}$ defined by

$$f_1(\vec{v}) = \phi_{\sigma^2 \mathbf{I}}(\mathbf{\Lambda}_{(1)}^{(1/2)} \vec{v}), \quad f_2(\vec{u}) = \det \left(\mathbf{M}_{\mathcal{I}} \left(\mathbf{\Lambda}_{(2)}^{(1/2)} \vec{u} \right) \right) \vec{V}_{\{N_n-n\}} \left(\mathbf{\Lambda}_{(2)}^{(1/2)} \vec{u} \right). \quad (3.12)$$

Then, $f_1 \in L^2(\mathbb{R}^n, \prod_{i=1}^n \phi(\vec{u}) d\vec{u})$, $\vec{u} \in \mathbb{R}^n$ and $f_2 \in L^2(\mathbb{R}^{N_n-n}, \prod_{i=1}^{N_n-n} \phi(\vec{u}) d\vec{u})$, $\vec{u} \in \mathbb{R}^{N_n-n}$. Write the Hermite expansions for f_1, f_2 :

$$\begin{aligned} f_1(\vec{u}) &= \sum_{a_1, \dots, a_n} d_{a_1 \dots a_n}^\sigma \prod_{i=1}^n H_{a_i}(u_i), \\ f_2(\vec{v}) &= \sum_{a_{n+1}, \dots, a_{N_n}} d_{a_{n+1} \dots a_{N_n}} \prod_{i=1}^{N_n-n} H_{a_{n+i}}(v_i). \end{aligned}$$

where

$$d_{a_1 \dots a_n}^\sigma = \frac{1}{a_1! \dots a_n!} \int_{\mathbb{R}^n} \phi_{\sigma^2 \mathbf{I}}(\mathbf{\Lambda}_{(1)}^{(1/2)} \vec{v}) \prod_{i=1}^n H_{a_i}(v_i) \phi(v_i) d\vec{v},$$

and

$$\begin{aligned} d_{a_{n+1} \dots a_{N_n}} &= \frac{1}{a_{n+1}! \dots a_{N_n}!} \int_{\mathbb{R}^{(N_n-n)}} \det \left(\mathbf{M}_{\mathcal{I}} \left(\mathbf{\Lambda}_{(2)}^{(1/2)} \vec{u} \right) \right) \vec{V}_{\{N_n-n\}} \left(\mathbf{\Lambda}_{(2)}^{(1/2)} \vec{u} \right) \\ &\quad \times \prod_{i=n+1}^{N_n} H_{a_i}(u_i) \phi(u_i) d\vec{u}. \end{aligned}$$

Then, with probability one, we have

$$f_1(\vec{Y}_{(1)})f_2(\vec{Y}_{(2)}) = \sum_{a_1, \dots, a_{N_n}} d_{a_1 \dots a_n}^\sigma d_{a_{n+1} \dots a_{N_n}} \prod_{i=1}^{N_n} H_{a_i}(Y_i).$$

Re-arranging the sum gives

$$\begin{aligned} f_1(\vec{Y}_{(1)})f_2(\vec{Y}_{(2)}) &= \sum_{q=0}^{\infty} \sum_{\{a_i\} \in \pi_n(q)} d_{a_1 \dots a_n}^\sigma d_{a_{n+1} \dots a_{N_n}} \prod_{i=1}^{N_n} H_{a_i}(Y_i) \\ &= \sum_{q=0}^{\infty} \sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}}^\sigma \tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) d\vec{s}, \end{aligned}$$

yielding

$$F_{(0,m)^n}^\sigma[X] \stackrel{a.s.}{=} \int_{(0,m)^n} \sum_{q=0}^{\infty} \sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}}^\sigma \tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) d\vec{s}.$$

To deduce the L^2 equality, write

$$A_Q = \int_{(0,m)^n} \sum_{q=0}^Q \sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}}^\sigma \tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) d\vec{s}.$$

Now note that the sequence $\{A_Q\}_{Q=1}^\infty$ is Cauchy. To prove this, note first that

$$\begin{aligned} \|A_{Q_1} - A_{Q_2}\|^2 &= \mathbb{E} \left[\int_{(0,m)^n} \sum_{q=Q_1}^{Q_2} \sum_{\pi_n(q)} d_{a_1 \dots a_n}^\sigma d_{a_{n+1} \dots a_{N_n}} \tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) d\vec{s} \right]^2 \\ &\leq m^n \int_{(0,m)^n} \mathbb{E} \left[\sum_{q=Q_1}^{Q_2} \sum_{\pi_n(q)} d_{a_1 \dots a_n}^\sigma d_{a_{n+1} \dots a_{N_n}} \tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) d\vec{s} \right]^2 \\ &\leq m^n \int_{(0,m)^n} \sum_{q=Q_1}^{Q_2} \mathbb{E} \left[\sum_{\pi_n(q)} d_{a_1 \dots a_n}^\sigma d_{a_{n+1} \dots a_{N_n}} \tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) d\vec{s} \right]^2. \end{aligned}$$

where the last inequality here follows from the orthogonality of spaces \mathcal{H}_q .

Exploiting the independence of the components of \vec{Y} , and applying a generalized Mehler's formula (see proof of proposition 2.1 in Estrade and León (2015) and Lemma 10.7 in Azaïs and Wschebor (2009)), we can bound the above expression by

$$m^{2n} \sum_{q=Q_1}^{Q_2} \sum_{\pi_n(q)} d_{a_1 \dots a_{N_n}}^{2,\sigma} a_1! \dots a_{N_n}!.$$

By convergence of the coefficients of the Hermite expansion, the above tends to zero when Q_1, Q_2 increase, and so we have that $\{A_Q\}_{Q=1}^\infty$ is Cauchy.

For the other faces, $J \in \{J\} \setminus T_n^\circ$ of dimension $d \equiv \dim(J) < n$, we have a slightly different expression for the coefficients d_a^σ in (3.9). Recall (2.42), from which it follows, similarly to the above, that the corresponding integrands are given by

$$f_1(\vec{v}) = \phi_{\sigma^2 \mathbf{I}_{n \times n}} \left(\vec{V}_{\mathcal{I}_J} \left(\Lambda_{(1)}^{(1/2)} \vec{v} \right) \right) \mathbb{1}_{\left\{ \langle \vec{V}_{\mathcal{I}_J} \left(\Lambda_{(1)}^{(1/2)} \vec{v} \right), \tilde{\eta}_J \rangle \geq 0 \right\}}, \quad (3.13)$$

$$f_2(\vec{u}) = \det \left(\mathbf{M}_{\mathcal{I}_J} \left(\Lambda_{(2)}^{(1/2)} \vec{u} \right) \right) \vec{V}_{\{N_n - n\}} \left(\Lambda_{(2)}^{(1/2)} \vec{Y}_{(2)} \right). \quad (3.14)$$

Thus, in terms of \vec{Y} we have that

$$\begin{aligned} d_{a|J}^\sigma &= \frac{1}{\vec{a}!} \int_{\mathbb{R}^n} \phi_{\sigma^2 \mathbf{I}_{k \times k}} \left(\vec{V}_{\mathcal{I}_J} \left(\Lambda_{(1)}^{(1/2)} \vec{v} \right) \right) \mathbb{1}_{\left\{ \langle \vec{V}_{\mathcal{I}_J} \left(\Lambda_{(1)}^{(1/2)} \vec{v} \right), \tilde{\eta}_J \rangle \geq 0 \right\}} \prod_{i=1}^n H_{a_i}(v_i) \phi(v_i) d\vec{v} \\ &\quad \times \int_{\mathbb{R}^{(N_k - k)}} \det \left(\mathbf{M}_{\mathcal{I}_J} \left(\Lambda_{(2)}^{(1/2)} \vec{u} \right) \right) \vec{V}_{\{N_n - n\}} \left(\Lambda_{(2)}^{(1/2)} \vec{u} \right) \prod_{i=1}^{N_k - k} H_{a_{n+i}}(u_i) \phi(u_i) d\vec{u}, \end{aligned} \quad (3.15)$$

where $\vec{a}! = a_1! \cdots a_{n+\frac{1}{2}k(k+1)}!$, and the second integral is evaluated over the $N_k - k$ coordinates of \vec{u} which appear in $\mathbf{M}_{\mathcal{I}_J} \left(\Lambda_{(2)}^{(1/2)} \vec{u} \right)$ and the one coordinate that appears in $\vec{V}_{\{N_n - n\}} \left(\Lambda_{(2)}^{(1/2)} \vec{u} \right)$. \square

Lemma 3.3. $F_{(0,m)^n}[X]$ admits the Wiener chaos expansion

$$F_{(0,m)^n}[X] \stackrel{L_2}{=} \sum_{q=1}^\infty \sum_{\pi_n(q)} d_{\vec{a}} \int_{(0,m)^n} \tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) d\vec{s}. \quad (3.16)$$

Alternatively,

$$F_{(0,m)^n}[X] \stackrel{L_2}{=} \sum_{q=1}^\infty I_q(f_q^m). \quad (3.17)$$

The variance, σ_m^2 , of $F_{(0,m)^n}[X]$ is given by

$$\sigma_m^2 = \sum_{q=1}^\infty \sum_{\vec{a} \in \pi_n(q)} \sum_{\vec{b} \in \pi_n(q)} d_{\vec{a}} d_{\vec{b}} \vec{a}! \vec{b}! R^m(\vec{a}, \vec{b}), \quad (3.18)$$

where $f_q^m \in \mathfrak{H}^{\odot q}$ is given by

$$f_q^m = \sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}} \int_{(0,m)^n} \text{symm} \left(\varphi_{\vec{s},1}^{\otimes a_1} \otimes \cdots \otimes \varphi_{\vec{s},N_n}^{\otimes a_{N_n}} \right) d\vec{s},$$

and the various coefficients are as follows:

$$d_{\vec{a}} = \frac{|\det(\Lambda_{(1)}^{(1/2)})|^{-(1/2)}}{\vec{a}!(2\pi)^{n/2}} \prod_{i=1}^n H_{a_i}(0) \\ \times \int_{\mathbb{R}^{(N_n-n)}} \det\left(\mathbf{M}_{\mathcal{I}}\left(\Lambda_{(2)}^{(1/2)}\vec{u}\right)\right) \vec{V}_{\{N_n+n\}}\left(\Lambda_{(2)}^{(1/2)}\vec{u}\right) \prod_{i=1}^{N_n-n} H_{a_{n+i}}(u_i) \phi(u_i) d\vec{u}, \quad (3.19)$$

$$R^m(\vec{a}, \vec{b}) = m^n \int_{(-m,m)^n} \sum_{\substack{d_{ij} \geq 0 \\ \sum_i d_{ij} = a_j \\ \sum_j d_{ij} = b_i}} \vec{a}! \vec{b}! \prod_{1 \leq i, j \leq N_n} \frac{(K_{ij}(\vec{s}))^{d_{ij}}}{(d_{ij})!} \prod_{1 \leq k \leq n} \left(1 - \frac{|\vec{s}_k|}{m}\right) d\vec{s}. \quad (3.20)$$

and

$$\sigma_m^2 = \sum_{q=1}^{\infty} u_q^m, \quad \text{and} \quad u_q^m = \sum_{\vec{a} \in \pi_n(q)} \sum_{\vec{b} \in \pi_n(q)} d_{\vec{a}} d_{\vec{b}} \vec{a}! \vec{b}! R^m(\vec{a}, \vec{b}), \quad (3.21)$$

Proof. By Lemma 3.1 and Lemma 3.2, it suffices to establish the L^2 convergence

$$\sum_{q=0}^{\infty} \sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}}^{\sigma} \int_{(0,m)^n} \tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) d\vec{s} \xrightarrow{\sigma \rightarrow 0} \sum_{q=0}^{\infty} \sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}} \int_{(0,m)^n} \tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) d\vec{s}.$$

It is straightforward that $\lim_{\sigma \rightarrow 0} d_{\vec{a}}^{\sigma} \rightarrow d_{\vec{a}}$, where

$$d_{\vec{a}} = \frac{|\det(\Lambda_{(1)}^{(1/2)})|^{-(1/2)}}{\vec{a}!(2\pi)^{n/2}} \prod_{i=1}^n H_{a_i}(0) \\ \times \int_{\mathbb{R}^{(N_n-n)}} \det\left(\mathbf{M}_{\mathcal{I}}\left(\Lambda_{(2)}^{(1/2)}\vec{u}\right)\right) \vec{V}_{\{N_n-n\}}\left(\Lambda_{(2)}^{(1/2)}\vec{u}\right) \prod_{i=1}^{N_n-n} H_{a_{n+i}}(u_i) \phi(u_i) d\vec{u}.$$

We start by showing that (3.16) is in L^2 . By Fatou's inequality

$$\mathbb{E} \left[\sum_{q=0}^Q \sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}} \int_{(0,m)^n} \tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) d\vec{s} \right]^2 \leq \lim_{\sigma \rightarrow 0} \mathbb{E} \left[\sum_{q=0}^Q \sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}}^{\sigma} \int_{(0,m)^n} \tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) d\vec{s} \right]^2 \\ = \lim_{\sigma \rightarrow 0} \mathbb{E} \sum_{q=0}^Q \left[\sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}}^{\sigma} \int_{(0,m)^n} \tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) d\vec{s} \right]^2,$$

the last line following from orthogonality.

Adding some positive terms to the sum and then using Lemma 3.1, the above is bounded by

$$\lim_{\sigma \rightarrow 0} \mathbb{E} \sum_{q=0}^{\infty} \left[\sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}}^{\sigma} \int_{(0,m)^n} \tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) d\vec{s} \right]^2 = \mathbb{E}[F_{(0,m)^n}[X]]^2 < \infty.$$

We introduce yet another shorthand notation, to be used for the remaining part of the current proof.

$$\tilde{I}_q^{\sigma} \triangleq \sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}}^{\sigma} \int_{(0,m)^n} \tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) d\vec{s} \quad \text{and} \quad \tilde{I}_q \triangleq \sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}} \int_{(0,m)^n} \tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) d\vec{s}.$$

With the above notation we have

$$F_{(0,m)^n}^{\sigma}[X] = \sum_{q=0}^{\infty} \tilde{I}_q^{\sigma} \quad \text{and} \quad F_{(0,m)^n}[X] = \sum_{q=0}^{\infty} \tilde{I}_q.$$

Since $\lim_{\sigma \rightarrow 0} d_{\vec{a}}^{\sigma} \rightarrow d_{\vec{a}}$, we have, for a fixed Q ,

$$\lim_{\sigma \rightarrow 0} \left\| \sum_{q=0}^Q \tilde{I}_q^{\sigma} \right\|_{L^2} = \left\| \sum_{q=0}^Q \tilde{I}_q \right\|_{L^2} = \sum_{q=0}^Q \|\tilde{I}_q\|_{L^2}.$$

Moreover, since $F_{(0,m)^n}[X], F_{(0,m)^n}^{\sigma}[X] \in L^2(\Omega)$, we have

$$\|F_{(0,m)^n}^{\sigma}[X]\|_{L^2} \xrightarrow{\sigma \rightarrow 0} \|F_{(0,m)^n}[X]\|_{L^2}. \quad (3.22)$$

Now,

$$\left\| \sum_{q=0}^{\infty} I_q - \sum_{q=0}^{\infty} I_q^{\sigma} \right\|_{L^2} \leq \left\| \sum_{q=0}^Q I_q - \sum_{q=0}^Q I_q^{\sigma} \right\|_{L^2} + \left\| \sum_{q=Q+1}^{\infty} I_q \right\|_{L^2} + \left\| \sum_{q=Q+1}^{\infty} I_q^{\sigma} \right\|_{L^2}.$$

Given $\varepsilon > 0$, we first choose Q' sufficiently large so that

$$\left\| \sum_{q=Q'+1}^{\infty} I_q \right\|_{L^2} < \varepsilon/3. \quad (3.23)$$

Consequently, because of (3.22), we can then choose σ sufficiently small so that

$$\left\| \sum_{q=Q'+1}^{\infty} I_q^{\sigma} \right\|_{L^2} < \varepsilon/3 \quad \text{and} \quad \left\| \sum_{q=0}^{Q'} I_q - \sum_{q=0}^{Q'} I_q^{\sigma} \right\|_{L^2} \leq \varepsilon/3. \quad (3.24)$$

Since $\|Y_i\| = 1$, we rely on the fundamental relation for the Wiener chaos

$$\prod_{i=1}^{N_n} H_{a_i}(Y_i(\vec{s})) = I_q \left(\text{symm}(\varphi_{\vec{s},1}^{\otimes a_1} \otimes \cdots \otimes \varphi_{\vec{s},N_n}^{\otimes a_{N_n}}) \right), \quad (3.25)$$

to write the expansion for $F_{(0,m)^n}[X]$, and then apply Fubini's theorem for multiple Wiener integrals to arrive at

$$\begin{aligned} F_{(0,m)^n}[X] &= \sum_{q=0}^{\infty} \sum_{\pi_n(q)} d_{a_1 \cdots a_{N_n}} \int_{(0,m)^n} \prod_{i=1}^{N_n} H_{a_i}(Y_i(\vec{s})) d\vec{s} \\ &= \sum_{q=0}^{\infty} I_q \left(\sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}} \int_{(0,m)^n} \text{symm} \left(\varphi_{\vec{s},1}^{\otimes a_1} \otimes \cdots \otimes \varphi_{\vec{s},N_n}^{\otimes a_{N_n}} \right) d\vec{s} \right) \\ &= \sum_{q=1}^{\infty} I_q(f_q^m), \end{aligned} \quad (3.26)$$

with

$$f_q^m(\lambda_1, \dots, \lambda_q) = \sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}} \int_{(0,m)^n} \text{symm} \left(\varphi_{\vec{s},1}^{\otimes a_1} \otimes \cdots \otimes \varphi_{\vec{s},N_n}^{\otimes a_{N_n}} \right) d\vec{s},$$

where $\lambda_1, \dots, \lambda_q \in \mathbb{M}$, and \mathbb{M} is as in (2.19).

Next, we proceed to calculate the variance, σ_m^2 . Using (3.16) and the orthogonality of \mathcal{H}_q for different q , we have

$$\begin{aligned} \sigma_m^2 &= \mathbb{E} \left[\sum_{q=0}^{\infty} \sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}} \int_{(0,m)^n} \tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) d\vec{s} \right]^2 \\ &= \sum_{q=0}^{\infty} \sum_{\vec{a} \in \pi_n(q)} \sum_{\vec{b} \in \pi_n(q)} d_{\vec{a}} d_{\vec{b}} \int \int_{(0,m)^{2n}} \mathbb{E} \left[\tilde{H}_{\vec{a}}(\vec{Y}_{\vec{s}}) \tilde{H}_{\vec{b}}(\vec{Y}_{\vec{u}}) \right] d\vec{s} d\vec{u}. \end{aligned} \quad (3.27)$$

By stationarity, this equals

$$\sum_{q=0}^{\infty} \sum_{\vec{a} \in \pi_n(q)} \sum_{\vec{b} \in \pi_n(q)} d_{\vec{a}} d_{\vec{b}} \int \int_{(0,m)^{2n}} \mathbb{E} \left[\prod_{i=1}^{N_n} H_{a_i}(Y_i(0)) \prod_{i=1}^{N_n} H_{b_i}(Y_i(\vec{u} - \vec{s})) \right] d\vec{s} d\vec{u}.$$

Then, a change in variables leads to

$$m^n \sum_{q=0}^{\infty} \sum_{\vec{a} \in \pi_n(q)} \sum_{\vec{b} \in \pi_n(q)} d_{\vec{a}} d_{\vec{b}} \int_{(-m,m)^n} \mathbb{E} \left[\prod_{i=1}^{N_n} H_{a_i}(Y_i(0)) \prod_{i=1}^{N_n} H_{b_i}(Y_i(\vec{v})) \right] \prod_{1 \leq k \leq n} \left(1 - \frac{|\nu_k|}{m} \right) d\vec{v}.$$

When $|\vec{a}| = |\vec{b}|$, we have [see Proposition 2.2.1 in Nourdin and Peccati (2012)]

$$\mathbb{E}\left[\prod_{i=1}^{N_n} H_{a_i}(Y_i(0)) \prod_{i=1}^{N_n} H_{b_i}(Y_i(\nu))\right] = \vec{a}! \vec{b}! \sum_{\substack{d_{ij} \geq 0 \\ \sum_i d_{ij} = a_j \\ \sum_j d_{ij} = b_i}} \prod_{1 \leq i, j \leq N_n} \frac{(K_{ij}(\vec{\nu}))^{d_{ij}}}{(d_{ij})!},$$

and zero otherwise. Thus, writing

$$R^m(\vec{a}, \vec{b}) = m^n \int_{(-m, m)^n} \vec{a}! \vec{b}! \sum_{\substack{d_{ij} \geq 0 \\ \sum_i d_{ij} = a_j \\ \sum_j d_{ij} = b_i}} \prod_{1 \leq i, j \leq N_n} \frac{(K_{ij}(\vec{\nu}))^{d_{ij}}}{(d_{ij})!} \prod_{1 \leq k \leq n} \left(1 - \frac{|\nu_k|}{m}\right) d\vec{\nu},$$

the variance is given by

$$\sigma_m^2 = \sum_{q=1}^{\infty} \sum_{\vec{a} \in \pi_n(q)} \sum_{\vec{b} \in \pi_n(q)} d_{\vec{a}} d_{\vec{b}} \vec{a}! \vec{b}! R^m(\vec{a}, \vec{b}) = \sum_{q=1}^{\infty} u_q^m,$$

where $u_q^m \geq 0$ is given by

$$u_q^m = \sum_{\vec{a} \in \pi_n(q)} \sum_{\vec{b} \in \pi_n(q)} d_{\vec{a}} d_{\vec{b}} \vec{a}! \vec{b}! R^m(\vec{a}, \vec{b}),$$

and we are done. \square

Lemma 3.4. *The coefficients in (3.16) satisfy*

$$\sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}}^2 \vec{a}! < C q^n. \quad (3.28)$$

Proof. The coefficients are

$$d_{a_1} \dots d_{a_n} = \frac{|\det(\Lambda_{(1)}^{(1/2)})|^{-(1/2)}}{(2\pi)^{n/2} a_1! \dots a_n!} \prod_{i=1}^n H_{a_i}(0),$$

$$d_{a_{n+1}} \dots d_{a_{N_n}} = \frac{1}{a_{n+1}! \dots a_{N_n}!} \int_{R^{N_n-n}} f_2(\vec{s}) \prod_{i=1}^{N_n-n} H_{a_{n+i}}(s_i) \phi(s_i) d\vec{s},$$

with f_2 as defined in (3.12). It is straightforward to see that $f_2(\vec{s})$ is a polynomial of degree $N_n - n + 1$ and thus has a finite Hermite polynomial expansion. This means that all the

terms $d_{a_{n+1}\dots a_{N_n}}$ with any of the indexes $a_i > N_n - n + 1$, $i \in \{(n+1), \dots, N_n\}$ are zero. Setting

$$C \triangleq |\det(\mathbf{\Lambda}_{(1)}^{(1/2)})|^{-(1/2)} \times \max_{a_{n+1}, \dots, a_{N_n}} (d_{a_{n+1}\dots a_{N_n}}^2 a_{n+1}! \dots a_{N_n}!),$$

gives

$$\sum_{\pi_n(q)} d_{a_1+\dots+a_{N_n}}^2 a_1! \dots a_{N_n}! \leq C \sum_{a_1, \dots, a_n \in \pi_n(q)} \left(\frac{H_{a_i}(0) \dots H_{a_n}(0)}{(2\pi)^{n/2} a_1! \dots a_n!} \right)^2 a_1! \dots a_n!.$$

Using Imkeller et al. (1995), Proposition 3, we have $\left| \frac{(H_{a_i}(0))^2}{a_i!} \right| \leq C$ and thus

$$\sum_{\pi_n(q)} d_{a_1+\dots+a_{N_n}}^2 a_1! \dots a_{N_n}! \leq C \sum_{a_1, \dots, a_n \in \pi_n(q)} 1 < C q^n.$$

For other faces $J \in \{J\} \setminus T_n^\circ$, by (2.42) the contribution to the Euler integral of face J is given by the limits of expressions of the form

$$\int_J \phi_{\sigma^2 \mathbf{I}}(\nabla X|_J(\vec{s})) \det(\nabla^2 X|_J(\vec{s})) X(\vec{s}) \mathbb{1}_{\{\langle \nabla X(\vec{s}), \vec{\eta}_J \rangle \geq 0\}} d\vec{s},$$

which are bounded by

$$\int_J \phi_{\sigma^2 \mathbf{I}}(\nabla X|_J(\vec{s})) \left(1 + (\det(\nabla^2 X|_J(\vec{s})))^2 \right) (1 + (X|_J(\vec{s}))^2) d\vec{s}.$$

Although, the functions in the bound are slightly different to the corresponding functions in the previous development for the contribution of T_n° , the remainder of the argument is essentially the same, and so we shall not write out the details. \square

3.2. Proof of Theorem 3.1

As previously shown in Lemma 3.1, the sum corresponding to the interior critical points can be written as the limit of

$$F_{(0,m)^n}^\sigma[X] = \int_{(0,m)^n} \phi_{\sigma^2 \mathbf{I}}(\nabla X(\vec{s})) \det(\nabla^2 X(\vec{s})) X(\vec{s}) d\vec{s}. \quad (3.29)$$

By Lemmas 3.1 and 3.3 we have the Wiener chaos representation of (3.29)

$$F_{(0,m)^n}[X] = \sum_{q=1}^{\infty} I_q(f_q^m). \quad (3.30)$$

We, therefore, need to establish a CLT for

$$\frac{1}{m^{n/2}} \sum_{q=1}^{\infty} I_q(f_q^m).$$

The CLT will follow immediately from Theorem 2.1 once we have checked that all the conditions of the theorem hold in our case.

We start with Condition (c) of Theorem 2.1. Conditions (a) and (b) will be deduced from our previous results and the proof of (d). As far as (c) is concerned, note firstly that, by Lemma 3.3, we have

$$f_q^m(\lambda_1, \dots, \lambda_q) = \sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}} \int_{(0,m)^n} \text{symm} \left(\varphi_{\vec{s},1}^{\otimes a_1} \otimes \dots \otimes \varphi_{\vec{s},N_n}^{\otimes a_{N_n}} \right) d\vec{s},$$

where

$$\text{symm} \left(\varphi_{\vec{s},1}^{\otimes a_1} \otimes \dots \otimes \varphi_{\vec{s},N_n}^{\otimes a_{N_n}} \right) = \frac{1}{q!} \sum_{\sigma} \varphi_{\vec{s},1}^{\otimes a_1} \otimes \dots \otimes \varphi_{\vec{s},N_n}^{\otimes a_{N_n}} (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(q)}).$$

Since $|\vec{a}| = q$, the inner sum can be written as

$$\sum_{j_1, \dots, j_q=1}^{N_n} c_{j_1, \dots, j_q} \varphi_{s,j_1} \otimes \dots \otimes \varphi_{s,j_q} \quad (3.31)$$

with the appropriate coefficients $\{c_{j_1, \dots, j_q}\}_{j_1, \dots, j_q=1}^{N_n}$, such that $c_{j_1, \dots, j_q} = 0$, whenever $\sum_i j_i \neq q$. Take $C(q) = \max_{j_1, \dots, j_q} \{c_{j_1, \dots, j_q}\}$. Using the above notation, we write

$$\begin{aligned} \frac{1}{m^n} (f_q^m \otimes_r f_q^m) &= \frac{1}{m^n} \sum_{\vec{a} \in \pi_n(q)} \sum_{\vec{b} \in \pi_n(q)} d_{\vec{a}} d_{\vec{b}} \\ &\times \int_{(0,m)^{2n}} \sum_{\substack{j_1, \dots, j_q \\ k_1, \dots, k_q}} c_{j_1, \dots, j_q} c_{k_1, \dots, k_q} (\varphi_{\vec{s},j_1} \otimes \dots \otimes \varphi_{\vec{s},j_q}) \otimes_r (\varphi_{\vec{u},k_1} \otimes \dots \otimes \varphi_{\vec{u},k_q}) d\vec{s} d\vec{u}. \end{aligned}$$

The following is true (see Section 2.5)

$$\begin{aligned} &(\varphi_{\vec{x},j_1} \otimes \dots \otimes \varphi_{\vec{x},j_q}) \otimes_r (\varphi_{\vec{y},k_1} \otimes \dots \otimes \varphi_{\vec{y},k_q}) \\ &= \left[\prod_{l=1}^r \langle \varphi_{\vec{x},j_l}, \varphi_{\vec{y},k_l} \rangle_{\mathfrak{H}} \right] \varphi_{\vec{x},j_{r+1}} \otimes \dots \otimes \varphi_{\vec{x},j_q} \otimes \varphi_{\vec{y},k_{r+1}} \otimes \dots \otimes \varphi_{\vec{y},k_q} \\ &= \prod_{l=1}^r K_{j_l, k_l}(\vec{x} - \vec{y}) \varphi_{\vec{x},j_{r+1}} \otimes \dots \otimes \varphi_{\vec{x},j_q} \otimes \varphi_{\vec{y},k_{r+1}} \otimes \dots \otimes \varphi_{\vec{y},k_q}, \end{aligned}$$

and

$$\begin{aligned} \langle \varphi_{\vec{x}, j_{r+1}} \otimes \cdots \otimes \varphi_{\vec{x}, j_q} \otimes \varphi_{\vec{y}, k_{r+1}} \otimes \cdots \otimes \varphi_{\vec{y}, k_q}, \varphi_{\vec{w}, i_{r+1}} \otimes \cdots \otimes \varphi_{\vec{w}, i_q} \otimes \varphi_{\vec{z}, m_{r+1}} \otimes \cdots \otimes \varphi_{\vec{z}, m_q} \rangle_{\mathfrak{H}^{\otimes 2q}} \\ = \prod_{l=1}^{q-r} K_{j_l, i_l}(\vec{x} - \vec{w}) \prod_{l=1}^{q-r} K_{k_l, m_l}(\vec{y} - \vec{z}). \end{aligned}$$

We have

$$\prod_{l=1}^r K_{j_l, k_l}(\vec{x} - \vec{y}) \leq \psi^r(\vec{x} - \vec{y}),$$

and

$$\prod_{l=1}^{q-r} K_{j_l, i_l}(\vec{x} - \vec{w}) \prod_{l=1}^{q-r} K_{k_l, m_l}(\vec{y} - \vec{z}) \leq \psi^{q-r}(\vec{x} - \vec{w}) \psi^{q-r}(\vec{y} - \vec{z}).$$

Since the number of summands in (3.31) is less than $(N_n)^q$, we have that

$$m^{-n} \|f_q^m \otimes_r f_q^m\|_{\mathfrak{H}^{\otimes 2q}}^2 = m^{-n} \langle f_q^m \otimes_r f_q^m, f_q^m \otimes_r f_q^m \rangle_{\mathfrak{H}} \leq m^{-n} \left(\sum_{\vec{a} \in \pi(q)} d_{\vec{a}}^2 \right) ((N_n)^q C(q))^2 \mathcal{Z}(t),$$

where

$$\mathcal{Z}(m) = \int_{(0, m)^{4n}} \psi^r(\vec{x} - \vec{y}) \psi^r(\vec{w} - \vec{z}) \psi^{q-r}(\vec{x} - \vec{w}) \psi^{q-r}(\vec{y} - \vec{z}) d\vec{x} d\vec{y} d\vec{w} d\vec{z}.$$

Next, using $\psi^r(\vec{x} - \vec{y}) \psi^{q-r}(\vec{x} - \vec{w}) \leq \psi^q(\vec{x} - \vec{y}) + \psi^q(\vec{x} - \vec{w})$ we have

$$\mathcal{Z}(m) \leq \mathcal{Z}_1(m) + \mathcal{Z}_2(m),$$

where

$$\mathcal{Z}_1(m) = \int_{(0, m)^{4n}} \psi^r(\vec{w} - \vec{z}) \psi^q(\vec{x} - \vec{w}) \psi^{q-r}(\vec{z} - \vec{y}) d\vec{x} d\vec{y} d\vec{w} d\vec{z},$$

$$\mathcal{Z}_2(m) = \int_{(0, m)^{4n}} \psi^r(\vec{w} - \vec{z}) \psi^q(\vec{x} - \vec{y}) \psi^{q-r}(\vec{z} - \vec{y}) d\vec{x} d\vec{y} d\vec{w} d\vec{z}.$$

Integrating with respect to \vec{x} and using

$$\int_{(0, m)^n} \psi^q(\vec{x} - \vec{w}) d\vec{x} \leq \int_{\mathbb{R}^n} \psi^q(\vec{v}) d\vec{v} < \infty,$$

then integrating with respect to the remaining coordinates and using

$$\begin{aligned} \int_{(0,m)^{3n}} \psi^r(\vec{w} - \vec{z}) \psi^{q-r}(\vec{z} - \vec{y}) * d\vec{y} d\vec{w} d\vec{z} = \\ = \int_{(-m,m)^{2n}} \psi^r \star \psi^{q-r}(\vec{w} - \vec{y}) d\vec{w} d\vec{y} \leq \int_{\mathbb{R}^n} \psi^r \star \psi^{q-r}(\vec{v}) d\vec{v} \end{aligned}$$

we get

$$\left(\sum_{\vec{a} \in \pi(q)} d_{\vec{a}}^2 \right) ((N_n)^q C)^2 \mathcal{Z}(m) < \infty,$$

from which it follows that

$$m^{-n} \|f_q^m \otimes_{q-p} f_q^m\|_{\mathfrak{H}^{\otimes 2p}} \leq \frac{C'}{m^n}.$$

Consequently

$$m^{-n} \|f_q^m \otimes_{q-p} f_q^m\|_{\mathfrak{H}^{\otimes 2p}} \xrightarrow{m \rightarrow \infty} 0$$

Thus, we have established that condition (c) of Theorem 2.1 is satisfied.

We now turn to the condition (d). We have to show that

$$\sup_{m \geq 1} m^{-n} \sum_{q=Q+1}^{\infty} q! \|f_q^m\|_{\mathfrak{H}^{\otimes q}}^2 \xrightarrow{Q \rightarrow \infty} 0.$$

The expression $m^{-n} \sum_{q=Q+1}^{\infty} q! \|f_q^m\|_{\mathfrak{H}^{\otimes q}}^2$ is the variance of the tail of the Wiener chaos expansion of $m^{-n} F_{(0,m)^n}[X]$. We need to show that it converges uniformly to zero. We have already developed this expansion in the previous results, with the only difference that now we have a normalization of m^{-n} ,

$$\begin{aligned} m^{-n} \sigma_m^2(Q) &= \sum_{q=Q}^{\infty} \sum_{\vec{a} \in \pi_n(q)} \sum_{\vec{b} \in \pi_n(q)} d_{\vec{a}} d_{\vec{b}} \\ &\quad \times \int_{(-m,m)^n} \mathbb{E} \left[\prod_{i=1}^{N_n} H_{a_i}(Y_i(0)) \prod_{i=1}^{N_n} H_{b_i}(Y_i(\vec{v})) \right] \prod_{1 \leq k \leq n} \left(1 - \frac{|\nu_k|}{m}\right) d\vec{v}. \end{aligned} \tag{3.32}$$

To show the uniform convergence we write an upper bound $C(Q)$ which is independent of m and vanishes as $Q \rightarrow \infty$. To construct such a bound we use Lemma 1 in Arcones (1994), which, for completeness, we reproduce.

Lemma 3.5 (Arcones (1994)). *Let \vec{V} and \vec{W} be two zero-mean Gaussian random vectors on \mathbb{R}^d , and assume that $\mathbb{E}V_iV_j = \mathbb{E}W_iW_j = \delta_{ij}$. Let $h : \mathbb{R}^d \rightarrow \mathbb{R}$ have Hermite rank r . (i.e. the lowest degree polynomial appearing in its Hermite expansion has degree r .) Write ψ_* for the supremum of the sum of the rows or columns of the covariance matrix $\text{Cov}(\vec{V}, \vec{W})$, and assume that $\psi_* < 1$. Then*

$$\left| \mathbb{E}[h(\vec{V}) - \mathbb{E}h(\vec{V})][h(\vec{W}) - \mathbb{E}h(\vec{W})] \right| \leq \psi_*^r \mathbb{E}h^2(\vec{V}).$$

Returning to the proof of Theorem 3.1, we now apply this lemma with

$$\vec{V} = (Y_1(0), \dots, Y_{N_n}(0)), \quad \vec{W} = (Y_1(\vec{\nu}), \dots, Y_{N_n}(\vec{\nu})),$$

and $h : \mathbb{R}^{N_n} \rightarrow \mathbb{R}$ given by

$$h(\vec{x}) = \prod_{i=1}^{N_n} H_{a_i}(x_i).$$

It is easy to check that

$$\psi_* \leq K^q \psi^q(\tau) \quad \text{and} \quad \mathbb{E}h^2 = \sum_{\pi_n(q)} d_a^2 a!.$$

Furthermore, since $q > 0$, $\mathbb{E} \prod_{i=1}^{N_n} H_{a_i}(Y_i) = 0$, and for $|\vec{\tau}|$ large enough, by the assumption on $\psi(\vec{\tau})$, we have that $K^q \psi^q(\tau) < 1$.

We now choose arbitrary $s \in \mathbb{R}^+$ and split the integral over two domains

$$\begin{aligned} & \int_{(-m, m)^n} \mathbb{E} \left[\prod_{i=1}^{N_n} H_{a_i}(Y_i(0)) \prod_{i=1}^{N_n} H_{b_i}(Y_i(\vec{\nu})) \right] \prod_{1 \leq k \leq n} \left(1 - \frac{|\nu_k|}{m} \right) d\vec{\nu} \\ &= \int_{R_0^n(s)} \mathbb{E} \left[\prod_{i=1}^{N_n} H_{a_i}(Y_i(0)) \prod_{i=1}^{N_n} H_{b_i}(Y_i(\vec{\nu})) \right] \prod_{1 \leq k \leq n} \left(1 - \frac{|\nu_k|}{m} \right) d\vec{\nu} \\ &+ \int_{(-m, m)^n \setminus R_0^n(s)} \mathbb{E} \left[\prod_{i=1}^{N_n} H_{a_i}(Y_i(0)) \prod_{i=1}^{N_n} H_{b_i}(Y_i(\nu)) \right] \prod_{1 \leq k \leq n} \left(1 - \frac{|\nu_k|}{m} \right) d\vec{\nu}. \end{aligned} \tag{3.33}$$

Here $R_0^n(s)$ is n -dimensional cube of side length s , centered at the origin.

For the first term corresponding to the integral over $R_0^n(s)$, we write

$$\sum_{q=Q}^{\infty} \int_{R_0^n(s)} \mathbb{E} \left[\sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}} \prod_{i=1}^{N_n} H_{a_i}(Y_i(0)) \sum_{\vec{b} \in \pi_n(q)} d_{\vec{b}} \prod_{i=1}^{N_n} H_{b_i}(Y_i(\vec{\nu})) \right] \prod_{1 \leq k \leq n} \left(1 - \frac{|\nu_k|}{m} \right) d\vec{\nu}.$$

Lemma 3.3 implies that the above sum is finite for all Q , and so it converges to zero as $Q \rightarrow \infty$. Regarding the second term, and reintroducing the summation from (3.32), consider

$$\sum_{q=Q}^{\infty} \int_{(-m,m)^n \setminus R_0^n(s)} \mathbb{E} \left[\sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}} \prod_{i=1}^{N_n} H_{a_i}(Y_i(0)) \sum_{\vec{b} \in \pi_n(q)} d_{\vec{b}} \prod_{i=1}^{N_n} H_{b_i}(Y_i(\vec{\nu})) \right] \times \prod_{1 \leq k \leq n} \left(1 - \frac{|\nu_k|}{m} \right) d\vec{\nu}. \quad (3.34)$$

By Lemma 3.5, we have

$$\left| \mathbb{E} \left[\sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}} \prod_{i=1}^{N_n} H_{a_i}(Y_i(0)) \sum_{\vec{b} \in \pi_n(q)} d_{\vec{b}} \prod_{i=1}^{N_n} H_{b_i}(Y_i(\vec{\nu})) \right] \right| \leq K^q \psi^q(\vec{\nu}) \sum_{\pi_n(q)} d_{\vec{a}}^2 \vec{a}!,$$

so we can bound the second integral by

$$\begin{aligned} \sum_{q=Q}^{\infty} \int_{(-m,m)^n \setminus R_0^n(s)} \left| K^q \psi^q(\vec{\nu}) \sum_{\pi_n(q)} d_{\vec{a}}^2 \vec{a}! \right| \prod_{1 \leq k \leq n} \left(1 - \frac{|\nu_k|}{m} \right) d\vec{\nu} \leq \\ \leq \sum_{q=Q}^{\infty} \int_{(-m,m)^n \setminus R_0^n(s)} K^q \psi^q(\vec{\nu}) \sum_{\pi_n(q)} d_{\vec{a}}^2 \vec{a}! d\vec{\nu}. \end{aligned}$$

By Lemma 3.4 we have $\sum_{\vec{a} \in \pi_n(q)} d_{\vec{a}}^2 \vec{a}! < Cq^n$, so that

$$\int_{(-m,m)^n \setminus R_0^n(s)} K^q \psi^q(\vec{\nu}) \left(\sum_{\pi_n(q)} d_{\vec{a}}^2 \vec{a}! \right) d\vec{\nu} \leq \int_{(-m,m)^n \setminus R_0^n(s)} K^q \psi^q(\vec{\nu}) Cq^n d\vec{\nu}.$$

Due to the assumption that $\psi(\vec{\nu}) \rightarrow 0$, we have, for $s \in \mathbb{R}^+$ large enough, that

$$\psi(\vec{\nu} \in \mathbb{R}^n \setminus R_0^n(s)) < \varepsilon < \frac{1}{K}, \quad (3.35)$$

which leads to the bound

$$\sum_{q=Q+1}^{\infty} Cq^n \frac{K^q \varepsilon^q}{\varepsilon} \int_{\mathbb{R}^n} \psi(\vec{\nu}) d\vec{\nu}.$$

Since $K\varepsilon < 1$ and

$$\int_{\mathbb{R}^n} \psi(\vec{\nu}) d\vec{\nu} < \infty,$$

the integral in (3.34) converges to zero uniformly in m . Overall, we conclude that $m^{-n}\sigma_m^2(Q) \xrightarrow{Q \rightarrow \infty} 0$ uniformly in m , which establishes that Condition (d) is satisfied.

To show (a) and (b), we note that we have already demonstrated that for sufficiently large s the integral over $(-m, m)^n \setminus R_0^n(s)$ converges to zero. Then,

$$R^m(\vec{a}, \vec{b}) \rightarrow R(\vec{a}, \vec{b}) = \int_{\mathbb{R}^n} \vec{a}! \vec{b}! \sum_{\substack{d_{ij} \geq 0 \\ \sum_i d_{ij} = a_j \\ \sum_j d_{ij} = b_i}} \prod_{1 \leq i, j \leq N_n} \frac{(K_{ij}(\vec{v}))^{d_{ij}}}{(d_{ij})!} d\vec{v},$$

leading to

$$\sigma_m^2 \xrightarrow{m \rightarrow \infty} \sigma_\Psi^2 \triangleq \sum_{q=1}^{\infty} u_q, \quad (3.36)$$

where

$$u_q = \sum_{\vec{a} \in \pi_n(q)} \sum_{\vec{b} \in \pi_n(q)} d_{\vec{a}} d_{\vec{b}} \vec{a}! \vec{b}! R(\vec{a}, \vec{b}).$$

This completes the proof of Theorem 3.1. \square

4. The mean value of upper Euler integral

In this section, we discuss the mean value of the Euler integral. As shown by Bobrowski and Borman (2012), the mean value of the Euler integral of Gaussian random field scales not by the volume of the domain of integration, as one would expect, but according to a one-dimensional measure of the domain. Specifically,

$$\mathbb{E}[\Psi_M[X]] = -\frac{\mathcal{L}_1^X(M)}{\sqrt{2\pi}} \quad (4.1)$$

where $\mathcal{L}_1^X(M)$ is the first Lipschitz-Killing curvature of M , as evaluated with respect to the metric induced by the random field X . (cf. Adler and Taylor (2007) for definitions.)

In this section, we re-establish this result by direct evaluation of the mean value through the Wiener chaos decomposition of the Euler integral. To do so, we make use of the next proposition, which can be proven using symmetry considerations.

Proposition 4.1. *Let X be tame Gaussian on \mathbb{R}^n , and $n > 1$. Then,*

$$\mathbb{E}[\det(\nabla^2 X)X] = 0. \quad (4.2)$$

Recall that in proving the CLT of the previous section we concentrated only on critical points in the interior of the parameter space which contributed to the Euler integral. Now, however, we need to consider all such points, since we are looking at an un-normalised mean, rather than an asymptotic limit.

What is now interesting, and very different to what we saw before, is that the chaos approach shows that none of the faces of dimension different from one can contribute to $\mathbb{E}[\Psi_M[X]]$, including the interior face. This gives, from this angle at least, some new intuition into the Bobrowski-Borman result.

To justify this claim, note that in our chaos expansions the mean values of random variables in a chaos of order greater than zero vanish, so that possible contribution to mean values may come only from the zeroth chaos. This, however, is characterized by the coefficients $d_{0\dots 0|J}$. Let $\{d_{\vec{a}}\}_{\vec{a}}$ be the coefficients of the chaos of some face J . In general, each $d_{\vec{a}}$ in the Wiener chaos expansion for J of dimension k factorizes as $d_{a_1\dots a_k}^{(1)} d_{a_{k+1}\dots a_{N(k)}}^{(2)}$, where

$$d_{a_1\dots a_k|J}^{(1)} = \lim_{\sigma \rightarrow 0} \frac{1}{\vec{a}!} \int_{\mathbb{R}^n} \phi_{\sigma^2 \mathbf{I}_{k \times k}} \left(\vec{V}_{\mathcal{I}_J} \left(\Lambda_{(1)}^{(1/2)} \vec{v} \right) \right) \mathbb{1}_{\left\{ \left\langle \vec{V}_{\mathcal{I}_J} \left(\Lambda_{(1)}^{(1/2)} \vec{v} \right), \tilde{\eta}_J \right\rangle \geq 0 \right\}} \times \prod_{i=1}^n H_{a_i}(v_i) \phi(v_i) d\vec{v}, \quad (4.3)$$

$$d_{a_{k+1}\dots a_{N(k)}|J}^{(2)} = \int_{\mathbb{R}^{(N_k-k)}} \det \left(\mathbf{M}_{\mathcal{I}_J} \left(\Lambda_{(2)}^{(1/2)} \vec{u} \right) \right) \vec{V}_{\{N_n-n\}} \left(\Lambda_{(2)}^{(1/2)} \vec{Y}_{(2)} \right) \times \prod_{i=1}^{N_k-k} H_{a_{n+i}}(u_i) \phi(u_i) d\vec{u}. \quad (4.4)$$

Note, we have that $d_{0\dots 0|J}^{(2)} = \mathbb{E}[\det(\nabla^2 X|_J) X|_J]$. Thus, by Proposition 4.1, the contribution to the mean value of the faces J , $\dim J > 1$ is zero. Since the underlying field is centered it is obvious that the zero dimensional faces, the vertexes, do not contribute to the mean as well. Overall, the conclusion is that only the edges contribute to the mean value of Euler integral.

To see what this implies in a simple example, take $M = T_n$ with additional assumption of isotropy. Then

$$d_{0,\dots,0}^{(2)} = \mathbb{E} \left[\frac{\partial^2 X(\vec{s})}{\partial s_i \partial s_i} X(\vec{s}) \right] = - \frac{\partial^2 \rho(0)}{\partial \tau_i \partial \tau_i} = -\lambda_2, \quad (4.5)$$

where λ_2 is the second spectral moment of X .

To calculate $d_{0,\dots,0}^{(1)}$, we consider

$$\phi_{\sigma I}(\nabla X|_J(\vec{s})) \det(\nabla^2 X|_J(\vec{s})) X(\vec{s}) \mathbb{1}_{\left\{ \left\langle \nabla X(\vec{s}), \tilde{\eta}_J \right\rangle \geq 0 \right\}}. \quad (4.6)$$

Having summed over all parallel edges and taking expectations, we can eliminate the indicator term $\mathbb{1}_{\{\langle \nabla X(\vec{s}), \eta_J \rangle \geq 0\}}$, since by stationarity we translate everything to the same range over $(0, m)$ which yields

$$\sum_{\{J_i | J_i \text{ parallel to } J\}} \mathbb{1}_{\{\langle \nabla X(\vec{s}), \vec{\eta}_{J_i} \rangle \geq 0\}} \equiv 1. \quad (4.7)$$

Consequently, by (3.19),

$$d_0^{(1)} = \frac{|\lambda_2|^{-(1/2)}}{(2\pi)^{1/2}}, \quad (4.8)$$

and since the set of the edges of T_n can be split into n families of parallel edges, we finally have

$$\mathbb{E}[\Psi_{[0,m]^n}[X]] = d_0^{(1)} \times d_0^{(2)} = -\frac{|\lambda_2|^{(1/2)} n \times m}{(2\pi)^{1/2}}. \quad (4.9)$$

This is precisely (4.1) for this case.

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